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A geometry for non-geometric string backgrounds

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Abstract: A geometric string solution has background fields in overlapping coordinate patches related by diffeomorphisms and gauge transformations, while for a non-geometric background this is generalised to allow transition functions involving duality transformations. Non-geometric string backgrounds arise from T-duals and mirrors of flux compactifications, from reductions with duality twists and from asymmetric orbifolds. Strings in 'T-fold' backgrounds with a local n-torus fibration and T-duality transition functions in $O(n,n;\mathbb{Z})$ are formulated in an enlarged space with a T^{2n} fibration which is geometric, with spacetime emerging locally from a choice of a T^n submanifold of each T^{2n} fibre, so that it is a subspace or brane embedded in the enlarged space. T-duality acts by changing to a different T^n subspace of T^{2n}. For a geometric background, the local choices of T^n fit together to give a spacetime which is a T^n bundle, while for non-geometric string backgrounds they do not fit together to form a manifold. In such cases spacetime geometry only makes sense locally, and the global structure involves the doubled geometry. For open strings, generalised D-branes wrap a T^n subspace of each T^{2n} fibre and the physical D-brane is the part of the physical space lying in the generalised D-brane subspace.

Keywords: Discrete and Finite Symmetries, D-branes, String Duality, Differential and Algebraic Geometry.

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1. Introduction

There is evidence that string theory can be consistently defined in non-geometric backgrounds in which the transition functions between coordinate patches involve not only diffeomorphisms and gauge transformations but also duality transformation [1]–[7]. Such backgrounds can arise from compactifications with duality twists [1, 2, 10, 11] or from acting on geometric backgrounds with fluxes with T-duality [3 − 5] or mirror symmetry [8, 9]. In special cases, the compactifications with duality twists are equivalent to asymmetric orbifolds [1] which can give consistent string backgrounds [12 − 14]. It is to be expected that transition functions involving duality transformations should arise since dualities are fundamental symmetries of the theory on the same footing as diffeomorphisms and gauge transformations. Not all such backgrounds are consistent in the quantum theory, and they must be restricted by demanding conformal, Lorentz and modular invariance on the world-sheet. Understanding how conventional geometry must be generalised to incorporate such backgrounds could provide important clues as to the nature of string theory or M-theory, as they require going beyond the familiar formulations of field theory in a geometric spacetime or a sigma-model with geometric target space.

Duality transformations sometimes take a geometric string background to a non-geometric one, as they can lead to transition functions between coordinate patches which are not the coordinate transformations of standard differential geometry but are exotic ones that involve dualities. One could take the point of view that in such circumstances there is an obstruction to T-dualizing and the background does not have a T-dual. However, the non-geometric configuration that follows from formally applying the T-duality rules seems to make sense as a string background. Locally, there is a conventional spacetime interpretation, and the problem is in the transition functions relating overlapping geometric neighbourhoods. A perturbative string background is specified by a conformal field theory (CFT), and a given CFT can arise from a number of different sigma-models with very different target space geometries. The different target space geometries can be thought of as different representations of the same CFT, and so it seems reasonable to consider situations with transition functions that relate different representations of the same CFT. In particular, T-duality changes the target space geometry to another giving the same CFT, and so T-duality transition functions should be allowed. Some examples (such as those related to asymmetric orbifolds) are known to give consistent backgrounds, and compactifying on a T-fold (as in [1, 2]) gives a conventional low-energy field theory. In this paper, it will be assumed that T-duality can be applied in such cases, and the properties of the resulting non-geometric backgrounds will be explored.

A geometric background (M, G, B, Φ) is a spacetime manifold M with metric G, 2-form gauge field B and scalar field Φ (possibly with other fields such as RR gauge fields) so that the transition functions in the overlaps between coordinate patches are diffeomorphisms, together with a gauge transformation so that $B' = B + d\Lambda$, i.e. $G, \Phi, H = dB$ are tensor fields and B is a gerbe connection on M. If M is a torus bundle with fibre T^n, then each transition function includes an element of $\mathrm{GL}(n,\mathbb{Z}) = \mathrm{SL}(n,\mathbb{Z}) \times \mathbb{Z}_2$, the group of large diffeomorphisms on T^n. However, string theory on such a background has a T-duality symmetry $\mathrm{O}(n,n;\mathbb{Z})$, and acting on the background with a T-duality transformation $g \in \mathrm{O}(n,n;\mathbb{Z})$ will take a transition function $S \in \mathrm{GL}(n,\mathbb{Z})$ to $S' = gSg^{-1}$ which will in general be in $\mathrm{O}(n,n;\mathbb{Z})$ and need not lie in the subgroup $\mathrm{GL}(n,\mathbb{Z})$ that acts through torus diffeomorphisms. This means that the new transition functions will in general mix momentum and winding modes, so that a geometrical configuration in one patch can translate into a stringy one featuring winding modes in the next. T-duality then indicates that transition functions in $\mathrm{O}(n,n;\mathbb{Z})$ should be allowed, and such configurations in general are not geometric backgrounds. The purpose here is to explore such non-geometrical backgrounds and present a formulation that captures the novel structure and allows T-duality to act. The full transition functions for the torus bundles that will be considered here are in $\mathrm{GL}(n,\mathbb{Z}) \ltimes \mathrm{U}(1)^n$, where each $\mathrm{U}(1)$ acts as a translation on a circle fibre.

This notion of what is meant by a geometric background could be broadened to allow transition functions that include integral shifts of the B-field, $B \to B + \Theta$ where $\Theta \in H^2(T^n, \mathbb{Z})$, so that B is a gerb-connection. These shifts are part of $\mathrm{O}(n,n;\mathbb{Z})$ and so such transition functions are included in the class of backgrounds considered here, and so one could choose to define the subset of backgrounds with classical geometries as either those with diffeomorphism transition functions, or to include those with B-twists. It will be convenient here to define the 'geometric subgroup' of $\mathrm{O}(n,n;\mathbb{Z})$ to be $\mathrm{GL}(n,\mathbb{Z})$ rather than the group $\mathrm{GL}(n,\mathbb{Z}) \ltimes \mathbb{Z}^{n(n-1)/2}$ with B-shifts.

It is convenient to combine the metric G_{ij} and 2-form B_{ij} on an n-torus into the $n\times n$ matrix $E_{ij} = G_{ij} + B_{ij}$. These moduli will be taken to be constant on T^n but allowed to depend on the coordinates of the base space N. A transition function in $\mathrm{GL}(n,\mathbb{Z})$ represented by an $n\times n$ matrix $h_i{}^j$ will relate the values of E in two patches in N by

$$E' = hEh^t \tag{1.1}$$

so that $G' = hGh^t$, $B' = hBh^t$. Consider now an element of $\mathrm{O}(n,n;\mathbb{Z})$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1.2}$$

where a, b, c, d are integer-valued $n\times n$ matrices, and such that g preserves the metric L

$$L = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{1.3}$$

The transformation of E under a T-duality transformation $g \in \mathrm{O}(n,n;\mathbb{Z})$ is non-linear [15–17]:

$$E' = (aE+b)(cE+d)^{-1}\,, \tag{1.4}$$

while the dilaton transforms as

$$e^{\Phi'} = e^{\Phi}\left(\frac{\det\, G'}{\det\, G}\right)^{1/4}\,. \tag{1.5}$$

A transition function in $\mathrm{O}(n,n;\mathbb{Z})$ then relates (E,Φ) and (E',Φ') in the overlap of two patches by the non-linear transformations (1.4), (1.5). The non-linear transformations mix G and B, and mix momentum modes on the torus with string winding modes, so that a geometric description in one patch can be translated to a non-geometric one in an adjacent patch.

Examples arise when there is a T-duality monodromy around a non-contractible loop. String theory compactified on T^n has $\mathrm{O}(n,n;\mathbb{Z})$ symmetry, and then compactification on a further circle can be twisted by an $\mathrm{O}(n,n;\mathbb{Z})$ monodromy, giving a stringy generalisation [1, 10] of a Scherk-Schwarz reduction [16]–[22]. If the monodromy is in the geometric $\mathrm{GL}(n;\mathbb{Z})$ subgroup that acts through T^n diffeomorphisms, then this can be lifted to a higher dimensional theory compactified on a T^n bundle over a circle, but for general $\mathrm{O}(n,n;\mathbb{Z})$ monodromy there is no such geometric interpretation. Locally there is a T^n fibration with G, B defined on an internal T^n, but there is no globally defined geometry on the bundle. If the circle coordinate is Y with the identification $Y \sim Y + 2\pi$, then the torus moduli G, B and dilaton satisfy the boundary conditions

$$\begin{aligned} E(Y+2\pi) &= \frac{aE(Y)+b}{cE(Y)+d} \\ e^{\phi(Y+2\pi)} &= e^{\phi(Y)}\left(\frac{\det\, G(Y+2\pi)}{\det\, G(Y)}\right)^{1/4}\,. \end{aligned} \tag{1.6}$$

A momentum mode at Y is identified with a linear combination of momentum and winding modes at $Y+2\pi$, so that the background is intrinsically stringy. A large class of such twisted reductions are equivalent to asymmetric orbifolds, and so are consistent string theory backgrounds. Modular invariance imposes an important constraint on such backgrounds, but those that arise as duality transforms of geometric backgrounds will be modular invariant.

Such twisted reductions arise as T-duals of flux compactifications. Consider for example reduction on a 3-torus with constant NS H-flux given by an integer m times the volume form.[1] A T-duality on any circle gives a twisted reduction on a T^2 bundle over a circle, with geometric monodromy

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \tag{1.7}$$

in $\mathrm{GL}(2,\mathbb{Z})$. Then a T-duality on one of the T^2 directions takes one back to the T^3 with flux [10], while the T-duality on the other fibre direction gives a background with monodromy in $\mathrm{O}(2,2,\mathbb{Z})$ given by [3],[5]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{1.8}$$

so that the transition functions are of the form (1.6).

As mirror symmetry is related to a T-duality on the fibres of Calabi-Yau spaces with a T^3 fibration [25], the mirror of a Calabi-Yau with NS flux on its T^3 fibres will in general be non-geometric of this type [9]. Similar examples can also arise with cosmic strings in 4 dimensions or p-branes in $D = p + 3$ dimensions with a duality monodromy around the string or brane. The general feature in all such examples are transition functions involving a duality symmetry. These duality symmetries could be T-dualities or U-dualities (e.g. in spaces with torus fibrations) or mirror symmetries (in spaces with Calabi-Yau fibrations). Backgrounds with T-duality transition functions will be referred to as T-folds, while those with S-duality, U-duality or mirror symmetry transition functions will be referred to as S-folds, U-folds or mirror-folds etc. Examples of S-fold compactification were investigated in [11]. In this paper, attention will be restricted to the case of T-folds, as these provide backgrounds that can be analysed in perturbative string theory.

For string theory compactified on an n-torus T^n, the internal momentum p^i combines naturally with the winding number w^i to form the momenta $p_L = p + w$, $p_R = p - w$ taking values in the Narain lattice. Conjugate coordinates X^i_L, X^i_R are needed e.g. to write vertex operators e^{kX_L}, e^{kX_R}. These are given in terms of the torus coordinates X^i and the coordinates $\widetilde{X}^i$ on a dual torus by $X_L = X + \widetilde{X}$, $X_R = X - \widetilde{X}$. For each i, if X^i is a coordinate on a circle of radius R_i, then $\widetilde{X}^i$ is the coordinate on the T-dual circle of

[1] This is not a conformal field theory as it stands. A consistent string background is obtained by allowing the moduli to depend on an extra coordinate, to obtain a space with topology $T^3 \times \mathbb{R}$. The product of this with 6-dimensional Minkowski space can be viewed as a NS 5-brane with transverse space $T^3 \times \mathbb{R}$ that is smeared over the T^3 so that the harmonic function in the ansatz is a function on $\mathbb{R}$ [10]. The discussion given here should be applied to such a conformal field theory.

radius $1/R_i$. The theory is then naturally formulated as a string theory with target space the doubled torus T^{2n} with coordinates $X, \widetilde{X}$, and the T-duality symmetry $\mathrm{O}(n,n;\mathbb{Z})$ acts naturally on T^{2n}. This doubles the degrees of freedom, but they can be halved again by imposing the constraints that X_L is left-moving and X_R is right moving. This can be written in terms of the pull-back world-sheet one-forms $dX = \partial_\alpha X d\sigma^\alpha$ as the self-duality conditions

$$dX_L = *dX_L\,, \qquad dX_R = - * dX_R\,, \tag{1.9}$$

where $*$ is the world-sheet Hodge dual.

The dual coordinates $\widetilde{X}$ are needed also for string field theory, as the string field $\Psi[X(\sigma)]$ for strings on a torus is a field $\Psi[X, \widetilde{X}, a, \tilde{a}]$ depending on the coordinates $X^i, \widetilde{X}^i$ as well as the Fourier coefficents $a^i_n, \tilde{a}^i_n$ [26]. This is usually expanded in terms of an infinite set of fields $\psi(X)$ on T^n, but could instead be expanded in terms of an infinite set of fields $\psi(X, \widetilde{X})$ on the doubled torus T^{2n}, and if there are transition functions that mix $X, \widetilde{X}$, then this latter formulation seems the more natural.

This suggests using a doubled formalism with a doubled torus T^{2n} containing both the original torus with coordinates X^i and a dual n-torus with coordinates $\widetilde{X}_i$. There is a natural $\mathrm{O}(n,n)$ invariant metric $ds^2_L = 2dX^i d\widetilde{X}_i$ on this space and the subgroup of the group $\mathrm{GL}(2n,\mathbb{Z})$ of large diffeomorphisms of T^{2n} that preserve this metric is $\mathrm{O}(n,n;\mathbb{Z})$. Then $\mathrm{O}(n,n;\mathbb{Z})$ transition functions between patches in the base space N are part of the data needed to construct a T^{2n} bundle over N with natural fibre metric ds^2_L. Thus the natural geometry that can be constructed from T-fold data is one in which the T^n bundle over N is replaced by a T^{2n} bundle over N which, as will be seen, can be viewed as a 'universal space' containing all possible T-duals of the original configuration. T-duality transition functions that glue momentum modes in one patch to linear combinations of momentum and winding modes in another patch then become geometric for the enlarged space in which the winding modes are represented geometrically on the dual n-torus, as the transition functions are now diffeomorphisms of T^{2n}. The transition functions for such a T^{2n} bundle are in $\mathrm{O}(n,n,\mathbb{Z}) \ltimes \mathrm{U}(1)^{2n}$. This doubling of course increases the degrees of freedom, and the right counting is obtained by imposing a covariant self-duality constraint that generalises (1.9) and which implies, roughly speaking, that half of the coordinates on the T^{2n} correspond to right-movers and half to left-movers, although the split into left and right moving chiral bosons depends on the position in N. The doubled formalism involving both the X and the dual coordinates $\widetilde{X}$ is essentially that of Cremmer, Julia, Liu and Pope [27], and related to those of [28, 29]. If the chiral boson constraint is implemented using the non-covariant formalism of [35], then the formulation presented here is closely related to that proposed by Tseytlin [30, 31] and used in [32], while the approach to chiral bosons of [37] leads to a formulation similar to that of [39].

The physical space is then a subspace $T^n \subset T^{2n}$, which might be thought of as a kind of n-brane in T^{2n}, and T-duality can be understood as changing which T^n subspace is the physical space. The T^n is a completely null subspace with respect to the $\mathrm{O}(n,n)$ metric i.e. all tangent vectors are null. Locally, over each coordinate patch U in N one can choose a local T^n slice of the T^{2n} bundle, and a conventional geometric picture arises on the local patch of spacetime which is a (trivial) T^n bundle over U. If these can be patched together to form a T^n bundle over N, then there is a geometric string background, but in general there will not be a global T^n slice of the T^{2n} bundle, and there is no way of picking out a spacetime that is a T^n bundle over N. Then the local spacetime patches (a trivial T^n bundle over U for each patch $U \subset N$) do not patch together to form a manifold, but instead form a T-fold.

The plan of the paper is as follows. In the next section 2 the doubled formalism is introduced in which the torus fibres are doubled from a T^n to a T^{2n} and string theory on a space which is locally a T^n bundle over each patch U in N is rewritten as a string theory on a space $\widetilde{M}$ which is a T^{2n} bundle over N subject to a self-duality constraint. This formulation is manifestly invariant under the T-duality group $\mathrm{O}(n,n;\mathbb{Z})$. Choosing a polarization, i.e. a local choice of $T^n \subset T^{2n}$ leads to the conventional formulation and to the standard Buscher T-duality transformations of the sigma-model geometry. In section 3, the $n = 1$ case is analysed and developed as an explicit example of the general constructions of later sections. In section 4, the action of T-duality is analysed and it is shown that it changes the polarization, i.e. it transforms the physical T^n subspace of T^{2n} to a different subspace, so that T-duality is the statement that the physics is independent of the choice of polarization. In section 5, polarizations are characterised geometrically in terms of product structures. In section 6, the topology of T-folds and their doubled formulation is carefully analysed, and it is seen that for a geometric background there is a global polarization, i.e. there is a T^n bundle over N arising as a subspace of the doubled space $\widetilde{M}$, and this subspace is the geometric spacetime. However, for non-geometric backgrounds, one can choose a polarization locally but the physical patches of spacetime do not patch together to form a manifold, so that there is a local notion of spacetime, but not a global one. In section 7, the formalism is extended to open strings and D-branes. As T-duality can take a Dp-brane to a Dp'-brane with $p \neq p'$, T-duality transition functions can glue a Dp-brane to a Dp'-brane, and it is shown how this naturally emerges from the formalism. In the final section 8, a number of further issues are discussed, including quantization, the generalisation to U-duality and the relationship with F-theory.

2. The duality-invariant action

2.1 The doubled formalism

In this section a formalism is presented for sigma-models whose target space M is locally a T^n bundle over a base space N. The T^n is considered as a subspace of a doubled torus T^{2n} containing both the original torus and a dual n-torus and an enlarged target space $\widetilde{M}$ is introduced which is a T^{2n} bundle over N. The discussion in this section is local, considering the spacetime as $U \times T^n$ for some coordinate patch U in N, embedded in an enlarged space $U \times T^{2n}$. The global issues involved in gluing such patches together will be analysed in section 6. Let Y^m be coordinates on (a patch U of) the base N and $\mathcal{P}^I = \mathcal{P}^I_\alpha d\sigma^\alpha$ $(I, J = 1, \ldots, 2n)$ be the covariant momenta on T^{2n} that are world-sheet one-forms and tangent vectors on T^{2n}. They satisfy the Bianchi identity

$$d\mathcal{P}^I = 0 \tag{2.1}$$

so that locally there are coordinates $\mathbb{X}^I$ with

$$\mathcal{P}^I_\alpha = \partial_\alpha \mathbb{X}^I \,. \tag{2.2}$$

Coordinates on T^{2n} will be chosen to satisfy periodicity conditions, so that $\mathbb{X}^I \in \mathbb{R}^{2n}/\mathcal{G}$ for some $2n$-dimensional lattice $\mathcal{G}$. This choice fixes the reparameterization invariance up to large diffeomorphisms, the group $\mathrm{GL}(2n,\mathbb{Z})$ of automorphisms of the lattice Γ. The world-sheet lagrangian is

$$\mathcal{L}_d = \frac{1}{2}\mathcal{H}_{IJ}\,\mathcal{P}^I \wedge *\mathcal{P}^J + \mathcal{P}^I \wedge *\mathcal{J}_I + \mathcal{L}(Y)\,, \tag{2.3}$$

where $\mathcal{H}_{IJ}(Y)$ is a positive-definite metric on T^{2n} that can depend on the base coordinates Y, $*$ denotes the world-sheet Hodge dual and $\mathcal{J}_I = \mathcal{J}_{I\alpha}(Y)d\sigma^\alpha$ are source terms that depend on Y but are independent of $\mathbb{X}$, while $\mathcal{L}(Y)$ is the remaining $\mathbb{X}$-independent part of the lagrangian. To simplify the discussion, the world-sheet will be taken to be flat for now, so that terms involving the curvature such as the Fradkin-Tseytlin term do not appear. As will be discussed in what follows, this theory is equivalent to a standard sigma-model formulation if the metric $\mathcal{H}$ is restricted to be a natural metric on the coset $\mathrm{O}(n,n)/\,\mathrm{O}(n)\times\mathrm{O}(n)$. The field equation from varying $\mathbb{X}$ is

$$d * (\mathcal{H}\mathcal{P} + \mathcal{J}) = 0\,. \tag{2.4}$$

The number of fibre coordinates has been doubled, so a self-duality constraint generalising (1.9) is imposed that halves the degrees of freedom. The constraint is

$$\mathcal{P}^I = L^{IJ} * (\mathcal{H}_{JK}\mathcal{P}^K + \mathcal{J}_J) \tag{2.5}$$

for some constant invertible symmetric matrix L^{IJ}. Then this together with the Bianchi identity (2.1) implies the field equation (2.4). Consistency of the constraint requires

$$S^2 = \mathbb{1}\,, \tag{2.6}$$

where

$$S^I{}_J = L^{IK}\mathcal{H}_{KJ} \tag{2.7}$$

and

$$\mathcal{H}L * \mathcal{J} = -\mathcal{J}\,. \tag{2.8}$$

Then $S = L\mathcal{H}$ has eigenvalues ± 1, and we choose L_{IJ} (the inverse of L^{IJ}) to be an $\mathrm{O}(n,n)$ invariant metric. (For heterotic strings, the metric L would be $\mathrm{O}(n,n+16)$ invariant.) For a flat target space, $\mathcal{H} = \mathbb{1}$ and the constraint reduces to (1.9).

The lagrangian is manifestly invariant under the rigid $\mathrm{GL}(2n,\mathbb{R})$ transformations

$$\mathcal{H} \to g^t\mathcal{H}g\,, \qquad \mathcal{P} \to g^{-1}\mathcal{P}, \qquad \mathcal{J} \to g^t\mathcal{J} \tag{2.9}$$

(with Y and $\mathcal{L}(Y)$ invariant). The corresponding transformation of the coordinates

$$\mathbb{X} \to g^{-1}\mathbb{X} \tag{2.10}$$

only preserves the boundary conditions if g is restricted to be in the subgroup $\mathrm{GL}(2n,\mathbb{Z}) \subset \mathrm{GL}(2n,\mathbb{R})$ preserving the lattice $\mathcal{G}$. The constraint (2.5) breaks $\mathrm{GL}(2n,\mathbb{R})$ to the subgroup $\mathrm{O}(n,n)$ preserving L^{IJ} and so breaks $\mathrm{GL}(2n,\mathbb{Z})$ to $\mathrm{O}(n,n;\mathbb{Z})$. Thus this formulation is manifestly invariant under the T-duality group $\mathrm{O}(n,n;\mathbb{Z})$.

To understand the significance of $\mathcal{J}$, it is useful to decompose it into a part proportional to dY and one proportional to $*dY$

$$\mathcal{J}_I = \mathcal{H}_{IJ}(\mathcal{A}^J + *\widetilde{\mathcal{A}}^J)\,, \tag{2.11}$$

where

$$\mathcal{A}^I = \mathcal{A}^I_m dY^m\,, \qquad \widetilde{\mathcal{A}}^I = \widetilde{\mathcal{A}}^I_m dY^m\,. \tag{2.12}$$

Then the constraint (2.8) becomes

$$S\mathcal{A} = -\widetilde{\mathcal{A}} \tag{2.13}$$

giving $\widetilde{A}$ in terms of A, while (2.5) becomes

$$(\mathcal{P} + \mathcal{A}) = S * (\mathcal{P} + \mathcal{A})\,. \tag{2.14}$$

The $\mathcal{A}^I_m$ are the components of a connection for the T^{2n} fibration over N, whose pull-backs to the world-sheet is $\mathcal{A}^I$, so that $\mathcal{P} + \mathcal{A}$ is a covariantized momentum, and the lagrangian could be written in terms of $\hat{\mathcal{P}} = \mathcal{P} + \mathcal{A}$ instead of $\mathcal{P}$ as

$$\mathcal{L}_d = \frac{1}{2}\mathcal{H}_{IJ}\,\hat{\mathcal{P}}^I \wedge *\hat{\mathcal{P}}^J + \mathcal{L}'(Y)\,. \tag{2.15}$$

The connection transforms as

$$\mathcal{A} \to g^{-1}\mathcal{A} \tag{2.16}$$

under $\mathrm{O}(n,n)$.

2.2 Right and left movers

As will be seen below, the doubled theory is equivalent to a standard sigma-model formulation provided the metric $\mathcal{H}_{IJ}$ is restricted to be a coset metric for $\mathrm{O}(n,n)/\,\mathrm{O}(n)\times\mathrm{O}(n)$, so that on identifying under the action of $\mathrm{O}(n,n;\mathbb{Z})$, the moduli space for such metrics is $\mathrm{O}(n,n;\mathbb{Z})\backslash\,\mathrm{O}(n,n)/\,\mathrm{O}(n)\times\mathrm{O}(n)$. It is convenient to parameterize the coset space $\mathrm{O}(n,n)/\,\mathrm{O}(n)\times\mathrm{O}(n)$ by a $2n\times 2n$ vielbein $\mathcal{V}^A{}_I(Y)$ which is an element of $\mathrm{O}(n,n)$ identified under the left action of $\mathrm{O}(n)\times\mathrm{O}(n)$, $\mathcal{V}\sim k\mathcal{V}$. The field $\mathcal{V}(Y)$ then transforms as

$$\mathcal{V} \to k(Y)\mathcal{V}g \tag{2.17}$$

under a rigid $\mathrm{O}(n,n)$ transformation $g\in\mathrm{O}(n,n)$ and a local $\mathrm{O}(n)\times\mathrm{O}(n)$ transformation $k(Y)\in\mathrm{O}(n)\times\mathrm{O}(n)$, so that $k^tk=\mathbb{1}$. The vielbein can be used to transform between the $\mathrm{O}(n,n)$ indices I,J and the $\mathrm{O}(n)\times\mathrm{O}(n)$ indices A,B. The metric is then

$$\mathcal{H} = \mathcal{V}^t\mathcal{V} \tag{2.18}$$

and is manifestly invariant under local $\mathrm{O}(n)\times\mathrm{O}(n)$ transformations and transforms under $\mathrm{O}(n,n)$ as in (2.9). The indices $A,B=1,\ldots,2n$ transform under $\mathrm{O}(n)\times\mathrm{O}(n)$ and can be split into indices $a,b=1,\ldots,n$ and $a',b'=1,\ldots,n$ for the two $\mathrm{O}(n)$ factors, $A=(a,a')$, so that in a natural basis

$$L^{AB}=\begin{pmatrix}L^{ab} & 0\\ 0 & L^{a'b'}\end{pmatrix}=\begin{pmatrix}\mathbb{1}^{ab} & 0\\ 0 & -\mathbb{1}^{a'b'}\end{pmatrix},\qquad S^{A}{}_{B}=\begin{pmatrix}\delta^{a}{}_{b} & 0\\ 0 & -\delta^{a'}{}_{b'}.\end{pmatrix} \tag{2.19}$$

Then

$$\mathcal{V}^{A}{}_{I}=\begin{pmatrix}\mathcal{V}^{a}{}_{I}\\ \mathcal{V}^{a'}{}_{I}\end{pmatrix},\qquad \mathcal{V}\mathcal{P}=\begin{pmatrix}\mathcal{P}^{a}\\ \mathcal{P}^{a'}\end{pmatrix}, \tag{2.20}$$

and

$$\mathcal{H}_{IJ}=\mathcal{V}^{a}{}_{I}\mathcal{V}^{b}{}_{J}\delta_{ab}+\mathcal{V}^{a'}{}_{I}\mathcal{V}^{b'}{}_{J}\delta_{a'b'}. \tag{2.21}$$

The self-duality constraint can now be written as

$$\begin{aligned}\mathcal{P}^{a}&=+*(\mathcal{P}^{a}+\mathcal{J}^{a})\\ \mathcal{P}^{a'}&=-*(\mathcal{P}^{a'}+\mathcal{J}^{a'})\end{aligned} \tag{2.22}$$

or

$$\begin{aligned}\hat{\mathcal{P}}^{a}&=+*\hat{\mathcal{P}}^{a}\\ \hat{\mathcal{P}}^{a'}&=-*\hat{\mathcal{P}}^{a'}\end{aligned} \tag{2.23}$$

Introducing null coordinates on the world-sheet $\alpha=(+,-)$ so that $\mathcal{P}_{\pm}=\pm*\mathcal{P}_{\pm}$, this becomes

$$\begin{aligned}\mathcal{P}^{a}_{-}&=-\frac{1}{2}\mathcal{J}^{a}_{-}, & \mathcal{J}^{a}_{+}&=0\\ \mathcal{P}^{a'}_{+}&=-\frac{1}{2}\mathcal{J}^{a'}_{+}, & \mathcal{J}^{a'}_{-}&=0\end{aligned} \tag{2.24}$$

or

$$\begin{aligned}\hat{\mathcal{P}}^{a}_{-}&=0\\ \hat{\mathcal{P}}^{a'}_{+}&=0\end{aligned} \tag{2.25}$$

In the case of a trivial bundle with constant $\mathcal{V}$ independent of Y and $\mathcal{J}=0$, this would imply $\partial_{-}X^{a}=0$ and $\partial_{+}X^{a'}=0$ so that X^{a} are right-movers and $X^{a'}$ are left-movers, giving the right count of degrees of freedom. More generally, the Y dependence of $\mathcal{V}$ means that the way the $\mathbb{X}^{I}$ are split into left-movers and right-movers depends on the position Y on the base, while the source $\mathcal{J}$ further modifies the constraint.

2.3 Choice of polarization

The physical spacetime is locally a product of an n-torus with a region of N, and the physical T^{n} is embedded in T^{2n}. In order to make contact with the conventional formulation, one needs to choose a polarization, i.e. to choose a splitting of T^{2n} into a physical T^{n} and a dual $\widetilde{T}^{n}$ for each point in N, splitting the fibre coordinates into the physical coordinates

$X \in T^n$ and the dual coordinates $\widetilde{X} \in \widetilde{T}^n$, and then write the theory in terms of the coordinates X alone, solving the constraint (2.5) to express $\widetilde{X}(\sigma)$ in terms of $X(\sigma)$. Then the variables X would be the ones integrated over in the functional integral and invariance of the theory under T-duality implies that the physics is independent of the choice of polarization.

In order to define a polarization or local product structure on the fibres, one first chooses a subgroup $\mathrm{GL}(n, \mathbb{R})$ of $\mathrm{O}(n, n)$ under which the fundamental $2n$ of $\mathrm{O}(n, n)$ splits into the fundamental representation n of $\mathrm{GL}(n, \mathbb{R})$ and the dual representation n', $2n \to n \oplus n'$. It will be useful to use a superscript i for the fundamental representation n (where $i = 1, ..., n$) and a subscript i for the dual representation n', and introduce constant projectors ${\Pi^i}_I$ and $\widetilde{\Pi}_{iI}$, so that

$$\mathcal{P} = \begin{pmatrix} {\Pi^i}_I \mathcal{P}^I \\ \widetilde{\Pi}_{iI} \mathcal{P}^I \end{pmatrix} = \begin{pmatrix} P^i \\ Q_i \end{pmatrix} . \tag{2.26}$$

This can be thought of as a choice of basis, but it is useful to introduce the projectors explicitly, so as to keep track of the choice of subgroup $\mathrm{GL}(n, \mathbb{R})$ of $\mathrm{O}(n, n)$; as we shall see, duality transformations change the projectors and change the subgroup $\mathrm{GL}(n, \mathbb{R})$ to a conjugate one. The norm of $\mathcal{P}$ with respect to the $\mathrm{O}(n, n)$ metric is $|\mathcal{P}|^2 = 2P^i Q_i$. The ${\Pi^i}_I$ projects the tangent space of T^{2n} onto a maximally isotropic or lagrangian subspace, i.e. a subspace which is null with respect to the metric L_{IJ} and of maximal dimension n, while $\widetilde{\Pi}_{iI}$ projects onto the complementary null subspace. This is equivalent to choosing a pure spinor for $Spin(n, n)$ [41]. Then

$$ {\Pi^i}_I {\Pi^j}_J L^{IJ} = 0 , \qquad \widetilde{\Pi}_{iI} \widetilde{\Pi}_{jJ} L^{IJ} = 0 , \tag{2.27}$$

and the metric L is off-diagonal in the $\mathrm{GL}(n)$ basis and can be written as

$$L = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{2.28}$$

The $\mathrm{GL}(n)$ subgroup is embedded in $\mathrm{O}(n, n)$ in this basis as matrices of the form

$$\begin{pmatrix} h^{-1} & 0 \\ 0 & h^t \end{pmatrix} , \tag{2.29}$$

where $h = {h^i}_j$ is an $n \times n$ matrix in $\mathrm{GL}(n)$.

Choosing a subgroup $\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{O}(n, n)$ allows a splitting of the tangent spaces. If the projectors in fact project onto a subgroup $\mathrm{GL}(n, \mathbb{Z}) \subset \mathrm{O}(n, n; \mathbb{Z})$, then this extends to a splitting of the coordinates $\mathbb{X}^I \to (X^i, \widetilde{X}_i)$ that is consistent with the boundary conditions, with $P^i = dX^i$, $Q_i = d\widetilde{X}_i$. Then

$$\mathbb{X} = \begin{pmatrix} {\Pi^i}_I \mathbb{X}^I \\ \widetilde{\Pi}_{iI} \mathbb{X}^I \end{pmatrix} = \begin{pmatrix} X^i \\ \widetilde{X}_i \end{pmatrix} , \tag{2.30}$$

with the X^i the coordinates of the T^n subspace and $\widetilde{X}_i$ the coordinates of the dual $\widetilde{T}^n$ subspace. The $\mathrm{O}(n, n)$-invariant metric in these coordinates is

$$ds^2 = 2dX^i d\widetilde{X}_i \tag{2.31}$$

so that the T^n submanifold with coordinates X^i is a maximally null subspace. Choosing a polarization splitting $T^{2n} \to T^n \oplus \widetilde{T}^n$ then corresponds to choosing a subgroup $\mathrm{GL}(n,\mathbb{Z}) \subset \mathrm{O}(n,n;\mathbb{Z})$.

In this basis the vielbein has components

$$\mathcal{V}^A{}_I = \begin{pmatrix} \mathcal{V}^a{}_i & \mathcal{V}^{aj} \\ \mathcal{V}^{a'}{}_i & \mathcal{V}^{a'j} \end{pmatrix} . \tag{2.32}$$

The local $\mathrm{O}(n) \times \mathrm{O}(n)$ symmetry can be used to choose a triangular gauge for $\mathcal{V}$, so that

$$\mathcal{V} = \begin{pmatrix} e^t & 0 \\ -e^{-1}B & e^{-1} \end{pmatrix} , \tag{2.33}$$

for some n-bein e_i^a and anti-symmetric $n \times n$ matrix B_{ij}. Then

$$\mathcal{H} = \mathcal{V}^t\mathcal{V} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} , \tag{2.34}$$

where the metric $G = e^t e$, i.e.

$$G_{ij} = e_i{}^a e_j{}^b \delta_{ab} . \tag{2.35}$$

As a result, the fibre metric $\mathcal{H}(Y)$ is parameterized by an $n \times n$ matrix $E(Y)$ given by

$$E_{ij} = G_{ij} + B_{ij} . \tag{2.36}$$

Note that the inverse of $\mathcal{H}$ is $\mathcal{H}^{IJ} = L^{IK}L^{JL}\mathcal{H}_{KL}$ so that

$$\mathcal{H}^{-1} = L\mathcal{H}L = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} , \tag{2.37}$$

and $(L\mathcal{H})^2 = \mathbb{1}$.

The current $\mathcal{J}$ decomposes into

$$J_i = \widetilde{\Pi}_{iI} L^{IJ} \mathcal{J}_J , \qquad K^i = \Pi^i{}_I L^{IJ} \mathcal{J}_J \tag{2.38}$$

so that in the $\mathrm{GL}(n)$ basis

$$\mathcal{J} = \begin{pmatrix} J_i \\ K^i \end{pmatrix} \tag{2.39}$$

and the source-term in the lagrangian becomes

$$\mathcal{P}^I \wedge *\mathcal{J}_I = P^i \wedge *J_i + Q_i \wedge *K^i . \tag{2.40}$$

The X^i are coordinates on a torus T^n and, as we shall see, the dynamics of X^i are governed by a sigma-model on T^n with metric G_{ij} and B-field B_{ij}, while the $\widetilde{X}_i$ are T-dual coordinates. If $B = 0$, the metric $\mathcal{H}$ restricted to on T^n is G_{ij} while restricted to $\widetilde{T}^n$ it is the inverse of this, G^{ij}, so that $\widetilde{T}^n$ is the dual torus to T^n with respect to the metric $\mathcal{H}$.

However, the $\mathcal{V}$ defined in (2.33) can be written as

$$\mathcal{V} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-1} \end{pmatrix} g(B), \qquad g(B) = \begin{pmatrix} \mathbb{1} & 0 \\ -B & \mathbb{1} \end{pmatrix} \tag{2.41}$$

in terms of the vielbein with $B = 0$, and acting on this with the $\mathrm{O}(n,n)$ element $g(B)$ generated by the 2-form B. In this way, the geometry with respect to $\mathcal{H}$ is the B-deformation of T^n and the dual torus $\widetilde{T}^n$ given by acting with $g(B)$.

2.4 Equivalence with standard formulation

One approach to solving the constraints is to solve the Bianchi identities $d\mathcal{P} = 0$ to obtain $2n$ coordinates $X^i, \widetilde{X}_i$. Instead, one can solve $dP = 0$ to obtain n coordinates X^i

$$P^i_\alpha = \partial_\alpha X^i \tag{2.42}$$

and then solve the self-duality constraint (2.5) to obtain $Q_i = \widetilde{Q}_i(P, *P, K, *K)$ where

$$\widetilde{Q}_i = G_{ij}(*P^j - K^j) + B_{ij}P^j \tag{2.43}$$

while (2.8) can be solved to give $J_i = \widetilde{J}_i$ where

$$\widetilde{J}_i = -G_{ij} * K^j + B_{ij}K^j\,. \tag{2.44}$$

Then by an argument similar to that given in [27], the field equations for both X and Y derived from the doubled lagrangian (2.3) together with the constraints (2.5), (2.8) are completely equivalent to those derived from the lagrangian

$$\mathcal{L}_s = \frac{1}{2}G_{ij}\, P^i \wedge *P^j + \frac{1}{2}B_{ij}\, P^i \wedge P^j - G_{ij}\, P^i \wedge K^j - \frac{1}{4}G_{ij}\, K^i \wedge *K^j + \mathcal{L}(Y)\,. \tag{2.45}$$

It is useful to decompose the world-sheet one-form

$$G_{ij}\, K^j = k_{im} * dY^m - v_{im}dY^m \tag{2.46}$$

in terms of some $k_{im}(Y)$, $v_{im}(Y)$. Then if $\mathcal{L}(Y)$ takes the form

$$\mathcal{L}(Y) = \frac{1}{2}\widetilde{G}_{mn}\, dY^m \wedge *dY^n + \frac{1}{2}\widetilde{B}_{mn}\, dY^m \wedge dY^n \tag{2.47}$$

for some $\widetilde{G}_{mn}(Y), \widetilde{B}_{mn}(Y)$ on the base manifold N, the full lagrangian is

$$\mathcal{L}(Z) = \frac{1}{2}\mathcal{G}_{PQ}\, dZ^P \wedge *dZ^Q + \frac{1}{2}\mathcal{B}_{PQ}\, dZ^P \wedge dZ^Q\,, \tag{2.48}$$

where $Z^P = (X^i, Y^m)$ are coordinates on the total space of a T^n bundle over N with

$$\begin{aligned} \mathcal{G}_{ij} &= G_{ij} \\ \mathcal{G}_{im} &= k_{im} \\ \mathcal{G}_{mn} &= \widetilde{G}_{mn} + \frac{1}{2}G^{ij}(k_{im}k_{jn} - v_{im}v_{jn}) \end{aligned} \tag{2.49}$$

and

$$\begin{aligned} \mathcal{B}_{ij} &= B_{ij} \\ \mathcal{B}_{im} &= v_{im} \\ \mathcal{B}_{mn} &= \widetilde{B}_{mn} + \frac{1}{2}G^{ij}(v_{im}k_{jn} - k_{im}v_{jn})\,. \end{aligned} \tag{2.50}$$

2.5 T-duality transformation rules

Consider an $\mathrm{O}(n,n)$ transformation by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.51}$$

where a, b, c, d are $n \times n$ matrices. This preserves the indefinite metric L, so that

$$g^t L g = L \quad \Rightarrow \quad a^t c + c^t a = 0\,, \quad b^t d + d^t b = 0\,, \qquad a^t d + c^t b = \mathbb{1}\,. \tag{2.52}$$

The transformation rules for $\mathcal{H}$ (2.9) give the non-linear transformation of E under a T-duality transformation $g \in \mathrm{O}(n,n)$ [16, 17]

$$E' = (aE + b)(cE + d)^{-1}\,. \tag{2.53}$$

The transformation (2.9) for $\mathcal{J}$ gives, using (2.39), $K' = cJ + dK$. Substituting for J using the solution $J = \widetilde{J}$ (2.44) of the constraint (2.8) gives

$$K'^i = (d^i{}_j + c^{ik} B_{kj}) K^j - c^{ik} G_{kj} * K^j\,. \tag{2.54}$$

This then implies the transformation rules for v, k via (2.46) and results in the standard T-duality transformation rules of [15−17], provided $\widetilde{G}_{mn}, \widetilde{B}_{mn}$ are invariant. This will be seen explicitly in the next section for $n = 1$.

3. The $n = 1$ case

The simplest example is that of $n = 1$, with a fibre that is an S^1. It is instructive to see how things work in this case explicitly. First, the formalism of the last section is applied to this case, and then topology and T-duality are discussed for this example to motivate the treatment of the general case in sections 4 and 6.

3.1 The doubled formalism for $n = 1$

For the case in which $n = 1$, the indices $i, j, \ldots$ all take the value 1, so that $B_{ij} = 0$ and $G_{11} = R^2$ where $R(Y)$ is the radius of the fibre circle with coordinate X^1, with the identification $X^1 \sim X^1 + 2\pi$. The 2-metric $\mathcal{H}$ on the doubled torus is simply

$$\mathcal{H} = \begin{pmatrix} R^2 & 0 \\ 0 & R^{-2} \end{pmatrix} \tag{3.1}$$

so that the lagrangian (2.3) becomes

$$\mathcal{L}_d = \frac{1}{2} R^2 dX \wedge *dX + \frac{1}{2} R^{-2} d\widetilde{X} \wedge *d\widetilde{X} + dX \wedge *J + d\widetilde{X} \wedge *K + \mathcal{L}(Y) \tag{3.2}$$

and the doubled torus consists of the X circle of radius R and the dual $\widetilde{X}$ circle of radius $\widetilde{R} = 1/R$.

Consider the polarization based on picking the X circle from T^2. The source term is given in terms of J, K with

$$R^2\, K = k_m * dY^m - v_m dY^m \tag{3.3}$$

and $J = \widetilde{J} = -R^2 * K$. The equivalent sigma-model is (2.48) with coordinates $Z^P = (X^1, Y^m)$ on the total space of an S^1 bundle over N with

$$\begin{aligned} \mathcal{G}_{11} &= R^2 \\ \mathcal{G}_{1m} &= k_m \\ \mathcal{G}_{mn} &= \widetilde{G}_{mn} + \frac{1}{2} R^{-2}(k_m k_n - v_m v_n) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \mathcal{B}_{11} &= 0 \\ \mathcal{B}_{1m} &= v_m \\ \mathcal{B}_{mn} &= \widetilde{B}_{mn} + \frac{1}{2} R^{-2}(v_m k_n - k_m v_n)\,. \end{aligned} \tag{3.5}$$

Consider the $\mathrm{O}(1,1;\mathbb{Z})$ transformation given by the action of

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

In the doubled formalism this takes $\mathbb{X} \to \mathbb{X}' = g^{-1}\mathbb{X}$ so that

$$X' = \widetilde{X}, \qquad \widetilde{X}' = X \tag{3.7}$$

exchanging the two dual circles, while $R' = 1/R$ and $J' = K$, $K' = J$. In the polarized form, the T-duality gives $R' = 1/R$ and $K' = \widetilde{J} = -R^2 * K$. This implies

$$K' = R'^{-2}(k'_m * dY^m - v'_m dY^m) = -k_m dY^m + v_m * dY^m \tag{3.8}$$

so that

$$k'_m = R^{-2} v_m \qquad v'_m = R^{-2} k_m\,. \tag{3.9}$$

This, together with the invariance of $\widetilde{G}, \widetilde{B}$, then gives the standard Buscher T-duality rules [15] for $\mathcal{G}, \mathcal{B}$

$$\begin{aligned} \mathcal{G}'_{11} &= \mathcal{G}_{11}^{-1} \\ \mathcal{G}'_{1m} &= \mathcal{G}_{11}^{-1}\mathcal{B}_{1m} \\ \mathcal{G}'_{mn} &= \mathcal{G}_{mn} - \mathcal{G}_{11}^{-1}(\mathcal{G}_{1m}\mathcal{G}_{1n} - \mathcal{B}_{1m}\mathcal{B}_{1n}) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \mathcal{B}'_{1m} &= \mathcal{G}_{11}^{-1}\mathcal{G}_{1m} \\ \mathcal{B}'_{mn} &= \mathcal{B}_{mn} - \mathcal{G}_{11}^{-1}(\mathcal{B}_{1m}\mathcal{G}_{1n} - \mathcal{G}_{1m}\mathcal{B}_{1n})\,. \end{aligned} \tag{3.11}$$

3.2 T-duality and transition functions for $n = 1$

Over a point $Y \in N$, there is a T^2 consisting of the X circle with radius $R(Y)$ and the $\widetilde{X}$ circle with radius $\widetilde{R}(Y) = 1/R$, and there is a choice of polarization, i.e. a choice of which one-cycle is part of the space-time and which is the dual circle. Suppose we choose the X circle to be the one that is part of the physical spacetime. T-duality acts by taking $R \to 1/R$, so that the physical X circle now has radius $1/R$ and the dual $\widetilde{X}$ one has radius R. However, there is an equivalent way of the viewing the T-duality transformation: one could keep the radius R fixed, but change the polarization so that it is now the $\widetilde{X}$ circle of radius $1/R$ that is part of the physical spacetime. The T-duality can then be viewed as an active transformation transforming the geometry of the T^2 with $R \to 1/R$, or as a passive transformation in which the T^2 geometry is kept fixed, but the choice of polarization is changed. In either case, the physical spacetime is changed from one with a circle of radius R to one with a circle of radius $1/R$. As the conformal field theory for a circle of radius R is the same as that for a circle of radius $1/R$, the change does not affect the physics, but changes the variables used to describe it.

Consider first the case in which the spacetime is a trivial bundle $S^1 \times N$, so that the doubled space is $T^2 \times N$. The physical spacetime is $S^1 \times N$ embedded in $T^2 \times N$ and T-duality acts to 'rotate' the S^1 within $T^2 = S^1_R \times S^1_{1/R}$ from the S^1_R of radius R to the $S^1_{1/R}$ of radius $1/R$. In the active view, the direction of the polarization is kept fixed but the T^2 is rotated, while in the passive view the T^2 is kept fixed but the polarization is rotated — the polarization is initially a vector pointing in the X direction in the $X, \widetilde{X}$ plane and the same effect is obtained by either rotating the vector through $\pi/2$ while keeping the $X, \widetilde{X}$ axes fixed, or by keeping the vector fixed and rotating the axes through $-\pi/2$.

Next consider a situation with a non-trivial transition function. Suppose U, U' are two coordinate patches in N with non-trivial intersection and that the corresponding patches of M are $U \times S^1_R$ with a circle of radius $R(Y)$ and $U' \times S^1_{R'}$ with a circle of radius $R'(Y)$. These can be glued together to form (part of) a manifold if $R(Y) = R'(Y)$ for $Y \in U \cap U'$. However, they can be glued together with a T-duality transition function to form (part of) a T-fold if $R'(Y) = 1/R(Y)$ for $Y \in U \cap U'$. The T-fold transition clearly does not make a smooth space as it involves gluing a large circle to a small one, but from the conformal field theory point of view it is the same CFT over U and over U' and the transition is simply a change of the variables used to parameterise the CFT, changing which of $\mathcal{P}^1, \mathcal{P}^2$ is to be viewed as a momentum and which is to be viewed as a winding number.

In the doubled formalism the two patches become $U \times S^1_R \times S^1_{1/R}$ and $U' \times S^1_{R'} \times S^1_{1/R'}$. In the manifold case with $R = R'$, these can be glued together in the natural way, gluing the X circle to the X' one, and the $\widetilde{X}$ circle to the $\widetilde{X}'$ one. However, for the T-fold case with $R' = 1/R$, there is now the possibility of making a manifold by gluing the X circle to the $\widetilde{X}'$ one as both have radius R, and gluing the $\widetilde{X}$ circle to the X' one as both have radius $1/R$. This fits together to make a smooth manifold $(U \cup U') \times T^2$, where over each point in $U \cup U'$ there is a circle of radius R and one of radius $1/R$. However, the physical spacetime has two patches, $U \times S^1_R$ over U with the X circle of radius R and $U' \times S^1_{R'}$ over U' with the X' circle of radius $R' = 1/R$, and clearly these patches do not fit together to form a submanifold of the doubled space $(U \cup U') \times T^2$. Thus the polarization changes by a T-duality in going from U to U'. There are local patches of spacetime in which there is a conventional picture of the physics in terms of strings moving in that spacetime, but there is no global spacetime. There is a global doubled space $(U \cup U') \times T^2$, but the physical subspace is defined locally and jumps discontinuously through T-duality transition functions.

This generalises to the case $n > 1$. In the next section, it will be seen that in general T-duality acts to change the polarization by changing the physical subspace $T^n \subset T^{2n}$, while in section 6 it will be seen that over each patch $U \subset N$ there is a local patch of spacetime $U \times T^n$ embedded in the doubled space $U \times T^{2n}$, but for T-folds these do not fit together to form a spacetime manifold, even though there is a doubled manifold which is a T^{2n} bundle over N.

4. T-duality symmetry

The doubled lagrangian $\mathcal{L}_d$ is invariant under the $\mathrm{O}(n,n)$ transformations (2.9),(2.10):

$$\mathcal{L}_d(\mathcal{H}', \mathcal{J}', \mathbb{X}') = \mathcal{L}_d(\mathcal{H}, \mathcal{J}, \mathbb{X})\,. \tag{4.1}$$

This means that $\mathcal{L}_d(\mathcal{H}, \mathcal{J}, \mathbb{X})$ and $\mathcal{L}_d(\mathcal{H}', \mathcal{J}', \mathbb{X})$ define the same theory, as they are related by a field redefinition $\mathbb{X} \to g\mathbb{X}$. Part of the specification of the theory is a choice of $(\mathcal{H}, \mathcal{J})$, and choices $(\mathcal{H}, \mathcal{J})$, $(\mathcal{H}', \mathcal{J}')$ related by $\mathrm{O}(n,n;\mathbb{Z})$ transformations give equivalent doubled theories.

A further piece of data needed to specify the theory is a choice of polarization, i.e. a choice of a projector $\Pi^i{}_I$ which selects a null submanifold T^n of the doubled space T^{2n} as the physical spacetime, together with the complementary projector $\widetilde{\Pi}_{iI}$ onto $\widetilde{T}^n$. Two projectors Π, Π' define the same T^n if they are related by a diffeomorphism of T^n, so that

$$\Pi' = h\Pi \tag{4.2}$$

for some $h^i{}_j \in \mathrm{GL}(n,\mathbb{Z})$. Then $(\mathcal{H}, \mathcal{J}, \Pi)$, $(\mathcal{H}', \mathcal{J}', \Pi')$ define equivalent theories provided that the projector is transformed as

$$\Pi' = h\Pi g \tag{4.3}$$

for some $h \in \mathrm{GL}(n,\mathbb{Z})$. That is, if one simultaneously transforms the geometric data $(\mathcal{H}, \mathcal{J})$ and the polarization Π, then nothing changes. As we shall now show, the conventional form of a T-duality transformation consists of keeping Π fixed and transforming $(\mathcal{H}, \mathcal{J})$ according to the Buscher rules. However, the same effect can be obtained by keeping $(\mathcal{H}, \mathcal{J})$ fixed and transforming the polarization Π. Thus T-duality can be thought of in two ways. In the first, the doubled torus is transformed, but the projection onto the physical subspace is kept fixed, while in the second the doubled torus is kept fixed, but the choice of physical subspace is changed.

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It will be useful to introduce the notation $\hat{I}$ for the $\mathrm{O}(n,n)$ indices in the $\mathrm{GL}(n)$ basis, so that for any vector v, $v^{\hat{I}} = (v^i, v_i)$ and the matrix giving the change from an arbitrary basis to the $\mathrm{GL}(n)$ basis is

$$\Phi^{\hat{I}}{}_J = \begin{pmatrix} \Pi^i{}_J \\ \widetilde{\Pi}_{iJ} \end{pmatrix} \tag{4.4}$$

with the corresponding matrix for the dual representation $\hat{\Phi} = L^{-1}\Phi L$ so that

$$\hat{\Phi}_{\hat{I}}{}^J = \begin{pmatrix} \widetilde{\Pi}_i{}^J \\ \Pi^{iJ} \end{pmatrix}, \tag{4.5}$$

where $\Pi^{iJ} = \Pi^i{}_I L^{IJ}$, $\widetilde{\Pi}_i{}^J = \widetilde{\Pi}_{iI} L^{IJ}$. The matrix $\Phi^{\hat{I}}{}_J$ can be thought of as a representative of the coset $\mathrm{O}(n,n)/\,\mathrm{GL}(n,\mathbb{R})$, or as a 'vielbein' converting $\mathrm{O}(n,n)$ indices to $\mathrm{GL}(n)$ ones. (The context should avoid confusion between the vielbein Φ and the dilaton.) It will also be useful to introduce

$$\mathcal{R}^{\hat{I}}{}_J = \begin{pmatrix} \Pi^i{}_J \\ -\widetilde{\Pi}_{iJ} \end{pmatrix} \tag{4.6}$$

so that in the $\mathrm{GL}(n)$ basis $\mathcal{R}^2 = \mathbb{1}$; this will play a useful role in the following sections.

Then the equations giving components in the $\mathrm{GL}(n)$ basis can be written as

$$\Phi\mathcal{P} = \begin{pmatrix} P^i \\ Q_i \end{pmatrix}, \qquad \Phi\mathcal{J} = \begin{pmatrix} J_i \\ K^i \end{pmatrix} \tag{4.7}$$

and

$$\hat{\Phi}\mathcal{H}\hat{\Phi}^t = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{4.8}$$

and this notation will help in following the effects of changes of polarization explicitly. In particular, (4.8) defines a metric G_{ij} and 2-form B_{ij} in terms of $\mathcal{H}$ and a polarization Φ

$$\begin{aligned} G^{ij} &= \Pi^{iI}\Pi^{jJ}\mathcal{H}_{IJ} \\ B_{ij}G^{jk} &= \widetilde{\Pi}_i{}^I \Pi^{kJ}\mathcal{H}_{IJ}\,. \end{aligned} \tag{4.9}$$

Similarly, K_i is defined from (4.7) by

$$K^i = \Pi^{iJ}\mathcal{J}_J\,. \tag{4.10}$$

The T-duality transformation rules $G \to G'$, $B \to B'$, $K \to K'$ (2.53), (2.54) are then obtained using the $\mathrm{O}(n,n)$ transformations for $\mathcal{H}, \mathcal{J}$ while keeping the polarization Φ fixed,

$$\mathcal{H} \to \mathcal{H}' = g^t\mathcal{H}g\,, \qquad \mathcal{J} \to \mathcal{J}' = g^t\mathcal{J}\,, \qquad \Phi \to \Phi' = \Phi \tag{4.11}$$

so that

$$\begin{aligned} G^{-1} &= \Pi\mathcal{H}\Pi^t & &\to & (G')^{-1} &= \Pi g^t\mathcal{H}g\Pi^t \\ BG^{-1} &= \widetilde{\Pi}\mathcal{H}\Pi^t & &\to & B'(G')^{-1} &= \widetilde{\Pi}g^t\mathcal{H}g\Pi^t \\ K &= \Pi\mathcal{J} & &\to & K' &= \Pi g^t\mathcal{J}\,. \end{aligned}$$

These same transformations $G \to G'$, $B \to B'$, $K \to K'$ can also be obtained by keeping $\mathcal{H}$ fixed while transforming Φ

$$\mathcal{H} \to \mathcal{H}' = \mathcal{H}\,, \qquad \mathcal{J} \to \mathcal{J}' = \mathcal{J}\,, \qquad \Phi \to \Phi' = \Phi g \tag{4.12}$$

so that

$$\Pi \to \Pi' = \Pi g \qquad \widetilde{\Pi} \to \widetilde{\Pi}' = \widetilde{\Pi} g\,. \tag{4.13}$$

Note that this could be supplemented with a $\mathrm{GL}(n)$ transformation, so that $\Pi \to \Pi' = h^{-1}\Pi g$, $\widetilde{\Pi} \to \widetilde{\Pi}' = (h^t)^{-1}\widetilde{\Pi} g$.

Thus the T-duality transformations can be viewed either as active transformations in which the geometry $\mathcal{H}, \mathcal{J}$ is changed while Π is kept fixed (4.11), or as a passive one in which the geometry $\mathcal{H}, \mathcal{J}$ is kept fixed but the polarization is changed (4.12), (4.13). In the latter viewpoint, the doubled geometry is unchanged, but the choice of physical subspace is transformed. The symmetry under T-duality is then the statement that the physics does not depend on the choice of physical subspace.

5. Product structures and polarizations

The polarization can be characterised in terms of a product structure on the T^{2n} fibres. A local product structure (sometimes called a real structure or pseudo-complex structure; see e.g. [42]) is a tensor $\mathcal{R}^I{}_J$ satisfying

$$\mathcal{R}^2 = \mathbb{1} \tag{5.1}$$

and also satisfying an integrability condition analogous to the vanishing of the Nijenhuis tensor for a complex structure. The local product structure allows the construction of two projectors, $\frac{1}{2}(\mathbb{1} \pm \mathcal{R})$. A product structure defines a splitting of the tangent space into two eigenspaces of $\mathcal{R}$ with eigenvalue ± 1. For a torus T^{2n}, this will extend to a splitting of the periodic torus coordinates into those of two T^n eigenspaces if the product structure is *integral*

$$\mathcal{R} \in \mathrm{GL}(2n, \mathbb{Z}) \tag{5.2}$$

so that it acts on the coordinates while preserving their periodicities.

A metric L_{IJ} is *pseudo-hermitean* with respect to $\mathcal{R}$ if

$$L_{IK}\mathcal{R}^K{}_J + L_{JK}\mathcal{R}^K{}_I = 0\,. \tag{5.3}$$

A choice of polarization of T^{2n} then corresponds to a choice of real structure with respect to which the $\mathrm{O}(n,n)$ metric L_{IJ} is pseudo-hermitean and which is integral, i.e. it satisfies (5.2). Then the projector Π onto the physical space is $\frac{1}{2}(\mathbb{1} + \mathcal{R})$ and the projector $\widetilde{\Pi}$ onto the dual space is $\frac{1}{2}(\mathbb{1} - \mathcal{R})$. A product structure and pseudo-hermitean $\mathrm{O}(n,n)$ invariant metric are together preserved by the subgroup $\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{O}(n,n)$, and for the transformations acting on the torus, it is preserved by $\mathrm{GL}(n, \mathbb{Z}) \subset \mathrm{O}(n,n;Z)$.

Under an active T-duality transformation $\mathcal{H}, \mathcal{J}$ change as in (4.11) but $\mathcal{R}$ does not, while under a passive T-duality transformation $\mathcal{H}, \mathcal{J}$ do not change but $\mathcal{R}$ transforms as

$$\mathcal{R} \to \mathcal{R}' = g^{-1}\mathcal{R} g\,. \tag{5.4}$$

A second local product structure is defined by

$$\mathcal{S}^I{}_J = L^{IK}\mathcal{H}_{KJ} \tag{5.5}$$

which satisfies $\mathcal{S}^2 = \mathbb{1}$, which follows from $\mathcal{H} = \mathcal{V}^t\mathcal{V}$ with $\mathcal{V} \in \mathrm{O}(n,n)$. This satisfies

$$L_{IK}\mathcal{S}^K{}_J - L_{JK}\mathcal{S}^K{}_I = 0\,. \tag{5.6}$$

The sign here implies the metric is not pseudo-hermitean with respect to $\mathcal{S}$, but that the metric is *compatible* with the product structure, so that locally the metric is a product metric. Note that also $\mathcal{H}$ is compatible with $\mathcal{S}$,

$$\mathcal{H}_{IK}\mathcal{S}^K{}_J - \mathcal{H}_{JK}\mathcal{S}^K{}_I = 0 \tag{5.7}$$

and the product

$$\mathcal{I} = \mathcal{S}\mathcal{R} \tag{5.8}$$

is a complex structure

$$\mathcal{I}^2 = -\mathbb{1} \tag{5.9}$$

with respect to which the metric L is hermitean

$$L_{IK}\mathcal{I}^K{}_J + L_{JK}\mathcal{I}^K{}_I = 0\,. \tag{5.10}$$

The three matrices q_a with $q_1 = \mathcal{R}, q_2 = \mathcal{S}, q_3 = \mathcal{I}$ satisfy the pseudo-quaternionic algebra

$$q_a q_b = f_{ab}{}^c q_c + \eta_{ab}\,, \tag{5.11}$$

where $a, b = 1, 2, 3$, $f_{ab}{}^c$ are the structure constants of $\mathrm{SL}(2,\mathbb{R})$ and $\eta_{ab} = diag(1, 1, -1)$ is the Cartan metric of $\mathrm{SL}(2,\mathbb{R})$. The commutation relations are those of the Lie algebra $\mathrm{SL}(2,\mathbb{R})$ and the subgroup of $\mathrm{O}(n,n)$ preserving these three structures is $\mathrm{O}(n)$ (the diagonal subgroup of $\mathrm{O}(n) \times \mathrm{O}(n)$).

A polarization requires an integral product structure $\mathcal{R}$ satisfying (5.2), which requires that $\mathcal{R}$ is constant, while $\mathcal{S}$ depends on the moduli $\mathcal{H}(Y)$ or $G(Y), B(Y)$, so that $\mathcal{S}, \mathcal{I}$ are tensors which depend on the coordinates Y, and $\mathcal{S}$ is not integral.

Then there are tensors $\mathcal{R}, L$ which are constant matrices in the adapted coordinate system used here, and in a suitable basis they take the form

$$\mathcal{R}^I{}_J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad L_{IJ} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{5.12}$$

In addition there are $\mathcal{H}_{IJ}, \mathcal{S}^I{}_J, \mathcal{I}^I{}_J$ which depend on Y. Choosing coordinates so that, at some point Y_0, $\mathcal{H}_{IJ}(Y_0) = \delta_{IJ}$, then at the point Y_0 in a suitable basis the $\mathcal{H}_{IJ}, \mathcal{S}^I{}_J, \mathcal{I}^I{}_J$ become

$$\mathcal{S}^I{}_J = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \mathcal{I}^I{}_J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \mathcal{H}_{IJ} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{5.13}$$

and it is straightforward to check that these matrices satisfy the algebraic conditions given above.

The projection $\Pi : T^{2n} \to T^n$ leads to a fibration of $\widetilde{M}$ over M in the case in which M is a geometric background, with $\Pi : (X, \widetilde{X}, Y) \to (X, Y)$. It also leads to a projection from the tangent bundle of T^{2n} to the sum of the tangent and cotangent bundles of T^n, $\Pi : T(T^{2n}) \to (T \oplus T^*)(T^n)$ with $\Pi : (X^i, \widetilde{X}_i, P^i, \widetilde{P}_i) \to (X^i, P^i, \widetilde{P}_i)$. In the tangent space $\mathbb{R}^{2n}$ at a point of T^{2n}, the choice of a projector Π is the choice of a subgroup $\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{O}(n, n)$, which is the choice of a maximal null subspace or maximally isotropic subspace, which corresponds to choosing a pure spinor of $Spin(n, n)$ for each point on T^{2n} [41]. Then projecting onto M, there is a pure spinor of $Spin(n, n)$ for each point in M, and so there is a real analogue of the generalised Calabi-Yau structure of [41] on the T^n fibres of M (preserved by $\mathrm{GL}(n) \subset \mathrm{O}(n, n)$ instead of by $\mathrm{U}(m, m) \subset \mathrm{O}(2m, 2m)$). Each of the structures q_a defines a generalised complex or real structure for the fibres T^n of M and together furnish what might be called a generalised pseudo-hyperkahler structure.

6. Topology

Let $\{U_\alpha\}$ be an open cover of the base N, $N = \cup_\alpha U_\alpha$,[2] and suppose that there is a set of transition functions $\widetilde{g}_{\alpha\beta}$ satisfying the usual compatibility relations

$$\begin{aligned} \widetilde{g}_{\alpha\alpha} &= \mathbb{1} && \text{in} \quad U_\alpha \\ \widetilde{g}_{\alpha\beta} &= \widetilde{g}^{-1}_{\beta\alpha} && \text{in} \quad U_\alpha \cap U_\beta \\ \widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha} &= \mathbb{1} && \text{in} \quad U_\alpha \cap U_\beta \cap U_\gamma \end{aligned} \tag{6.1}$$

If the transition functions are in $\mathrm{GL}(2n; \mathbb{Z}) \ltimes \mathrm{U}(1)^{2n}$ they can be used to construct a space $\widetilde{M}$ as a T^{2n} bundle over N. The coordinates $\mathbb{X}^I$ on T^{2n} take values in $\mathbb{R}^{2n}/\Gamma$ where Γ is a lattice with automorphism group $\mathrm{GL}(2n; \mathbb{Z})$. Then the coordinates in $U_\alpha \times T^{2n}$ are $(Y_\alpha, \mathbb{X}^I_\alpha)$ and in $U_\alpha \cap U_\beta$ the T^{2n} fibre coordinates are related by a matrix $(g_{\alpha\beta})^I{}_J$ in $\mathrm{GL}(2n; \mathbb{Z})$ and a shift $x_{\alpha\beta}$ in $\mathrm{U}(1)^{2n}$

$$\mathbb{X}_\alpha = g^{-1}_{\alpha\beta}\mathbb{X}_\beta + x_{\alpha\beta} \tag{6.2}$$

and the transformations for $\mathcal{P}, \mathcal{H}, \mathcal{J}, L$ in overlapping patches given by

$$\mathcal{P}_\alpha = g^{-1}_{\alpha\beta}\mathcal{P}_\beta\,, \qquad L_\alpha = g^{-1}_{\alpha\beta}L_\beta(g^{-1}_{\alpha\beta})^t \tag{6.3}$$

and

$$\mathcal{H}_\alpha = g^t_{\alpha\beta}\mathcal{H}_\beta g_{\alpha\beta}\,, \qquad \mathcal{J}_\alpha = g^t_{\alpha\beta}\mathcal{J}_\beta \tag{6.4}$$

are those following from the T-duality rules. The transition functions can be thought of as defining a change of coordinates on the doubled torus in transforming from $U_\alpha \times T^{2n}$ to $U_\beta \times T^{2n}$, and these transformations of $\mathcal{P}, \mathcal{H}, \mathcal{J}, L$ are those needed for them to be well-defined objects on $\widetilde{M}$; such objects will be referred to as *tensors* on the doubled bundle.

For the special class of bundles for which $g_{\alpha\beta} \in \mathrm{O}(n, n; \mathbb{Z}) \subset \mathrm{GL}(2n; \mathbb{Z})$, $L_\alpha = L_\beta$ and there is a constant metric L_{IJ} of signature (n, n). The positive definite metric $\mathcal{H}$ depends on Y in general and will transform non-trivially between patches. If N is orientable, the total

[2] In this section α, β will label coordinate patches and not world-sheet coordinates.

space $\widetilde{M}$ will be orientable only if the structure group is in $\mathrm{SL}(2n;\mathbb{Z})$, which for invariant L requires the structure group to be in $\mathrm{SO}(n,n;\mathbb{Z})$. For structure group $\mathrm{O}(n,n;\mathbb{Z})$, $\widetilde{M}$ will be non-orientable in general.

Next we wish to discuss a polarization defined by a product structure $\mathcal{R}$ satisfying (5.1)–(5.3). Suppose such a structure $\widetilde{\mathcal{R}}_\alpha$ satisfying (5.1)–(5.3), is introduced for each patch U_α. As it is integral, $\widetilde{\mathcal{R}}_\alpha$ must be constant over U_α, but might be different in different patches ($\widetilde{\mathcal{R}}_\alpha \neq \widetilde{\mathcal{R}}_\beta$ for $\alpha \neq \beta$). Then $\widetilde{\mathcal{R}}_\alpha$ defines a splitting of the fibres over U_α, $T^{2n} \to T^n \oplus \widetilde{T}^n$, selecting a T^n and a dual $\widetilde{T}^n$. The $\widetilde{\mathcal{R}}_\alpha$ will form a tensor if they are related by the geometric transition functions

$$\widetilde{\mathcal{R}}_\alpha = g_{\alpha\beta}^{-1}\widetilde{\mathcal{R}}_\beta g_{\alpha\beta}\,. \tag{6.5}$$

If this is the case, the T^n subspaces of the fibres selected by restricting to the $+1$ eigenspace of $\mathcal{R}$ fit together to form a T^n bundle over N, as do the dual tori $\widetilde{T}^n$. Given a choice of T^n subspace over a patch U_{α_0} corresponding to a product structure $\widetilde{\mathcal{R}}_{\alpha_0}$, then defining $\widetilde{\mathcal{R}}$ in all other patches through (6.5) gives a choice of T^n for all other patches and gives a T^n bundle over N. The subgroup of $\mathrm{O}(n,n;\mathbb{Z})$ preserving a product structure (with a pseudo-hermitean $\mathrm{O}(n,n)$ metric) is $\mathrm{GL}(n,\mathbb{Z})$, so that if the structure group of the bundle is $\mathrm{GL}(n,\mathbb{Z})$, then $\widetilde{\mathcal{R}}_\alpha = \widetilde{\mathcal{R}}_\beta$.

The theory is specified by a choice of $(\mathcal{H},\mathcal{J},\mathcal{R})$ for each patch, and if all three are tensors with geometric transition functions (6.4), (6.5), then the polarization selects a physical subspace $U_\alpha \times T^n$ for each α and these fit together to form a T^n bundle over N. However, such geometric transition functions are not T-duality transition functions. Recall that a T-duality corresponds to transforming $(\mathcal{H},\mathcal{J})$ while keeping $\mathcal{R}$ fixed, or equivalently to transforming $\mathcal{R}$ while keeping $(\mathcal{H},\mathcal{J})$ fixed. The construction here involves a bundle with $(\mathcal{H},\mathcal{J})$ glued together with $\mathrm{O}(n,n;\mathbb{Z})$ transition functions (6.4), so that for the transition functions to be T-dualities requires a constant fibre product structure with trivial transition functions

$$\mathcal{R}_\alpha = \mathcal{R}_\beta\,. \tag{6.6}$$

Then the T-fold is a bundle with $\mathrm{O}(n,n;\mathbb{Z})$ transition functions $g_{\alpha\beta}$ with $(\mathcal{H},\mathcal{J},\mathcal{R})$ glued using (6.4), (6.6), and for each α, this selects a physical spacetime patch $U_\alpha \times T^n$.

A constant product structure (6.6) will be consistent with the geometric transition functions (6.5) only if the transition funcitons $g_{\alpha\beta}$ are all in the geometric subgroup $\mathrm{GL}(n,\mathbb{Z}) \subset \mathrm{O}(n,n;\mathbb{Z})$ preserving $\mathcal{R}$. Then for 'geometric' bundles with $\mathrm{GL}(n,\mathbb{Z})$ structure group, the physical subspace selected by a polarization corresponding to a constant product structure (6.6) is a T^n bundle over N, and the local patches $U_\alpha \times T^n$ fit together to form a spacetime manifold, while for $\mathrm{O}(n,n;\mathbb{Z})$ structure group they do not.

Any spacetime point is in a local spacetime patch $U_\alpha \times T^n$ and the local physics is described by strings moving in this spacetime patch. For a geometrical background, these fit together to form a manifold which is a T^n bundle over N, but for a non-geometric background or a T-fold these local patches do not fit together to form a manifold. For a geometric background there is a global polarization, i.e. there is a well-defined way of choosing a physical spacetime inside the doubled space, while for a non-geometric one a physical spacetime can only be found locally, but a global one does not exist.

This can also be thought of from a 'passive' viewpoint. Given a choice of polarization corresponding to a product structure $\mathcal{R}$, choose a patch U_{α_0}, say, and introduce a product structure $\widetilde{\mathcal{R}}_{\alpha_0} = \mathcal{R}$ in this patch and extend $\widetilde{\mathcal{R}}$ to all patches using the rule (6.5) to define a tensor $\widetilde{\mathcal{R}}_\alpha$, and there will be $g_{\alpha_0\beta} \in \mathrm{O}(n,n;\mathbb{Z})$ such that

$$\widetilde{\mathcal{R}}_\beta = g^{-1}_{\beta\alpha_0} \widetilde{\mathcal{R}}_{\alpha_0} g_{\beta\alpha_0} \,. \tag{6.7}$$

Then $\widetilde{\mathcal{R}}_\alpha$ defines a T^n bundle over N, and selects a reference T^n over each patch U_α. Then the physical subspace is defined by $\mathcal{R}$ and is obtained by an $\mathrm{O}(n,n;\mathbb{Z})$ transformation of the reference T^n subspace, as in U_β

$$\mathcal{R} = g_{\beta\alpha_0} \widetilde{\mathcal{R}}_\beta g^{-1}_{\beta\alpha_0} \,. \tag{6.8}$$

Over each U_α, there is a physical space $U_\alpha \times T^n$ embedded in $U_\alpha \times T^{2n}$ which can be thought of as a piece of the physical spacetime. If the transition functions are geometric (in the appropriate $\mathrm{GL}(n;\mathbb{Z})$ subgroup), these pieces fit together to form a spacetime which is a T^n bundle over N, but more generally the local pieces of spacetime do not fit together to form a manifold.

T-duality acts on these bundles by an $\mathrm{O}(n,n;\mathbb{Z})$ transformation g which is constant, so that it is the same for each patch α. It takes the physical subspace defined by a product structure $\mathcal{R}$ to the physical subspace corresponding to $\mathcal{R}' = g^{-1}\mathcal{R}g$ while taking the group $\mathrm{GL}(n,\mathbb{Z})$ of elements U preserving $\mathcal{R}$ to a conjugate $\mathrm{GL}(n,\mathbb{Z})$ subgroup of $\mathrm{O}(n,n;\mathbb{Z})$ consisting of elements gUg^{-1}. In this way it can change the maximal null $T^n \subset T^{2n}$ to any other maximal null T^n subspace and, as T-duality is a symmetry, different choices of polarization give the same physics, but in T-dual representations.

7. Open strings and D-branes

The doubled formalism can be used to discuss open strings in M also. Consider first the case of a trivial bundle with $M = N \times T^n$. If, for some i, $X^i(\sigma^\alpha)$ is an open string coordinate satisfying Neumann boundary conditions $\partial_n X^i = 0$ at a world-sheet boundary, then the T-dual coordinate $\widetilde{X}_i$ will satisfy Dirichlet boundary conditions $\partial_t \widetilde{X}_i = 0$, where ∂_n and ∂_t are the world-sheet derivatives normal and tangential to the boundary, respectively. Conversely, if X^i is Dirichlet, $\widetilde{X}_i$ will be Neumann. Then the $2n$ doubled coordinates $\mathbb{X}^I$ split into n Neumann directions $\mathbb{X}_N$ and n Dirichlet directions $\mathbb{X}_D$. The Dirichlet directions form a maximal null T^n submanifold.

In addition, a polarization gives a splitting of the $\mathbb{X}^I$ into n physical coordinates X and n dual coordinates $\widetilde{X}$, and in general this will be different from the splitting according to boundary conditions. Then the n physical coordinates X split into p_X Dirichlet coordinates X_D and $n - p_X$ Neumann ones X_N for some p_X, while the dual coordinates $\widetilde{X}$ split into $n - p_X$ Dirichlet coordinates $\widetilde{X}_D$ and p_X Neumann ones $\widetilde{X}_N$, so that the n Dirichlet coordinates are $\mathbb{X}_D = (X_D, \widetilde{X}_D)$ and the n Neumann coordinates are $\mathbb{X}_N = (X_N, \widetilde{X}_N)$. In the physical T^n, there is a D-brane wrapping the T^{p_X} with coordinates X_D. If p_Y of the coordinates of N satisfy Dirichlet boundary conditions, then there is a Dp-brane with $p = p_X + p_Y$ wrapping a T^{p_X} submanifold of the internal T^n.

In the doubled formalism, there is a Dp-brane with $p = p_Y + n$ wrapping the T^n submanifold (with coordinates $\mathbb{X}_D$) of the internal T^{2n}, and how much of this D-brane is visible in the physical spacetime depends on how many of the $p_Y + n$ Dirichlet directions are in the physical slice. A T-duality will change the physical polarization to a different $T^n \subset T^{2n}$ so that the dimension of the intersection with the Dirichlet T^n will change, and the number p_X of physical Dirichlet directions will also change. For example, for a spacetime with a T^2 fibration, $n = 2$, the doubled fibre coordinates consist of two Dirichlet directions $\mathbb{X}_D^1, \mathbb{X}_D^2$ and two Neumann ones $\mathbb{X}_N^1, \mathbb{X}_N^2$. If the physical polarization consists of the two Dirichlet directions $X = (\mathbb{X}_D^1, \mathbb{X}_D^2)$, this represents a $D(p_Y + 2)$ brane wrapped on the internal T^2, while a T-duality in the $\mathbb{X}_D^2$ direction will give a physical spacetime with coordinates $\mathbb{X}_D^1, \mathbb{X}_N^2, Y$ with a $D(p_Y + 1)$ brane wrapping the $\mathbb{X}_D^1$ direction. A further T-duality gives a physical spacetime with coordinates $\mathbb{X}_N^1, \mathbb{X}_N^2, Y$ and there is a Dp_Y brane that doesn't wrap the internal torus.

Consider now the interacting case, with lagrangian (2.3). The doubled torus coordinates $\mathbb{X}^I$ are to be split into coordinates satisfying Neumann boundary conditions on the edge of the world-sheet, and coordinates satisfying Dirichlet ones. Introducing a Dirichlet projector $(\Pi_D)^I{}_J$ and a Neumann projector $(\Pi_N)_I{}^J$ the boundary conditions are

$$\Pi_D \partial_t \mathbb{X} = 0 \tag{7.1}$$

and

$$\Pi_N(\mathcal{H}\partial_n \mathbb{X} + \mathcal{J}_n) = 0 \tag{7.2}$$

on the boundary. Varying the action with lagrangian (2.3) gives a boundary term that will vanish if

$$\Pi_D \Pi_N^t = 0\,. \tag{7.3}$$

The boundary conditions are consistent with the constraint (2.5) only if

$$\Pi_D L = L \Pi_N \tag{7.4}$$

(with $L = L^{IJ}$ the inverse metric). It then follows that there is a suitable choice of coordinates such that

$$L^{IJ} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad (\Pi_D)^I{}_J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad (\Pi_N)_I{}^J = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{7.5}$$

The boundary conditions in N will be the standard ones, corresponding to a Dp_Y-bane embedded in N, for some p_Y.

The Dirichlet coordinates $\mathbb{X}_D = \Pi_D \mathbb{X}$ are then coordinates for a maximal null T^n within T^{2n}. For each patch $U_\alpha \subset N$, there will be Dirichlet and Neumann projectors $(\Pi_D)_\alpha, (\Pi_N)_\alpha$ and so there will be a corresponding local product structure $(\hat{\mathcal{R}}_\alpha)^I{}_J$. Consistency of the boundary conditions requires that the $\hat{\mathcal{R}}_\alpha$ satisfy a condition of the form (6.5) so that the Dirichlet fibres fit together to make a T^n bundle over N. In the doubled picture, one could say that there is a D-brane wrapping the T^n fibres of this Dirichlet bundle. Over each patch U_α there is a T^{2n}, and two T^n subspaces, the Dirichlet T^n (parameterised by

$\mathbb{X}_D$, with complement the Neumann T^n) and the physical T^n (parameterised by X, with complement the dual $\widetilde{T}^n$), and those physical directions X_D which are also Dirichlet will be where the physical D-brane is said to be. The intersection of the Dirichlet T^n and physical T^n will be a T^{p_X} (parameterised by X_D) and there is a physical D-brane wrapping the T^{p_X} subspace of the physical T^n, so that there is a $D(p_X+p_Y)$-brane in the patch of the physical space $U_\alpha \times T^n$. A T-duality will change the physical T^n subspace and so change p_X. The transition function between two intersecting patches on N will be a T-duality changing the physical space and hence the value of p_X in general, unless the transition functions are in the geometric $\mathrm{GL}(n;\mathbb{Z})$. Thus for a T-fold, a D$p$-brane in one region can become a Dp'-brane in another with $p' \neq p$. For example, for a twisted reduction with T-duality monodromy, taking a Dp-brane round the reduction circle will bring it back transformed by the T-duality.

8. Discussion

For the theories discussed here, at each point in N there is an internal CFT with $2n$ currents $\mathcal{P}^I$ and associated conserved charges. The CFT can be represented by a sigma-model with a torus T^n as target space, in which n of the charges are interpreted as momenta on the T^n and the remaining n are interpreted as string winding charges. However, this is not unique as the same CFT can be represented in terms of different tori, depending on how the $\mathcal{P}$-charges are split into momenta and string charges. Locally, one can pick a polarization of the $\mathcal{P}$, and so choose which of the n-torus subspaces is to be viewed as the internal part of the spacetime. Often there will be a natural choice of torus in which all the radii are large compared to the string length so that the low-energy field theory description is useful. However, for T-folds, there is no global choice of polarization and so over different regions in the base space N there will be different choices of torus representation of the CFT. In such cases, there is typically no representation that has a low-energy description as a field theory on the total space, but on dimensionally reducing on the internal space, one obtains a low-energy description as a field theory on the base N.

For each region on N, some of the $\mathcal{P}^I$ are geometrical and realised as momenta, but the subset of the $\mathcal{P}^I$ that is geometrical can change from region to region, and each of the $\mathcal{P}^I$ is potentially geometrical. In the doubled formalism, all are treated as geometrical and correspond to momenta on the doubled torus T^{2n} so that the duality is a manifest symmetry, and the conventional description arises from choosing a T^n subspace that is to be interpreted as part of the physical spacetime. The duality twists in the transition functions mean that there is no way to separate the spacetime momenta from string winding states, and so both must be treated on the same footing, and the doubled formalism provides a natural way of doing so.

The discussion has been largely classical and it is important to extend this to the quantum theory. First the theory should be coupled to a curved world-sheet and the Fradkin-Tseytlin coupling of the world-sheet curvature to the dilaton $\Phi(Y)$ is added (the dilaton does not depend on $\mathbb{X}$ as T-duality requires isometries in the torus directions). The conventional approach to quantization would be to choose a polarization of $\mathbb{X}$ into

$X, \widetilde{X}$, solve the constraint (2.5) to express $\widetilde{X}$ in terms of X and so obtain a sigma-model formulation which can be quantized in the conventional way, with consistency requiring conformal invariance, and the change in the functional measure under T-duality gives the dilaton transformation (1.5). However, if there is not a global way of choosing a polarization, then there is an obstruction to doing this globally. One approach would be to use this polarized form locally and try and take into account the T-duality transition functions, but this would make many features of the theory obscure.

It seems natural to use instead the duality-covariant doubled formulation, integrating over the $\mathbb{X}$. In this approach the difficulty lies in implementing the self-duality constraint (2.5) in the quantum theory. The constraint (2.5) says, roughly speaking, that half of the $\mathbb{X}$ are right-movers and half are left-movers. There are several approaches that have been proposed for the quantization of chiral bosons [32]–[37] and one could try one of these. For example, adopting the approach of [35] to chiral bosons gives Tseytlin's formulation [30, 31], while the approach of [37] leads to the formulation of [39]. Including the coupling of the dilaton to the world-sheet curvature, the T-duality transformation of the dilaton arises from the transformation of the functional measure [30, 31]. The resulting theory will in general have anomalies in both the world-sheet conformal and Lorentz symmetries, and the vanishing of both of these imposes 'field equations' restricting the background geometry. (For example, in certain cases this would lead to separate beta-function equations for the conformal field theory of the left-movers and for that of the right-movers [39].) An additional constraint follows from requiring modular invariance. Although there are problems with writing down the partition function for a single chiral boson [40], the full theory here is required to have a well-defined modular invariant partition function.

An alternative approach is through gauged sigma-models. The world-sheet current j^I_α defined by

$$j^I = \mathcal{P}^I - L^{IJ} * \left(\mathcal{H}_{JK}\mathcal{P}^K + \mathcal{J}_J\right) \tag{8.1}$$

is conserved classically and instead of constraining it to vanish, one can gauge the corresponding $\mathrm{U}(1)^n$ symmetry through coupling to world-sheet gauge fields $A_{\alpha I}$, giving a formulation closely related to that of [16]. Again, the T-duality transformation of the dilaton follows from the transformation of the functional measure.

It is straightforward to supersymmetrise the doubled formulation presented here. This will be discussed elsewhere, where the conditions for extended world-sheet supersymmetry lead to complex geometries similar to those of [42, 43] and related to the generalised complex structures of [41].

Here spaces with T-duality transition functions were considered but this clearly generalises to transition functions involving other dualities. For M-theory in a background with a T^n fibration there is a U-duality symmetry [44] and so one should allow spaces with transition functions in the U-duality group $E_n(\mathbb{Z})$. Such spaces arise from U-duality transforms of geometric backgrounds with fluxes or of T-folds, or from reductions with U-duality twists. There are clearly relations to the F-theory approach of [6, 7]. For backgrounds which have a Calabi-Yau fibration, one could allow transition functions that are mirror transforms, to give mirror-folds. Indeed, when the mirror symmetry can be understood as a T-duality, these would be examples of T-folds. One could have patches in which there are different string theories provided that the transition functions involve the dualities relating the different string theories. For example one could have IIA string theory on a patch $U_\alpha \times S^1$ and IIB string theory on an overlapping patch $U_\beta \times S^1$ provided the transition function is a T-duality on the S^1, or IIA string theory in a neighbourhood with a K3 fibration patched onto a heterotic string theory in a a neighbourhood with a T^4 fibration.

Consider M-theory backgrounds with a T^n fibration. These correspond to IIA backgrounds with a T^{n-1} fibration, so that the doubled formulation should involve the coordinates X^i of the T^{n-1} and the coordinates $\widetilde{X}_i$ conjugate to the string winding charge w^i on the T^{n-1}. In the M-theory picture, the string winding modes become membrane wrapping modes and the string charge w^i becomes a membrane charge $Z^{i\,11}$ for a membrane wrapping the i'th direction on the T^{n-1} and the M-theory circle with coordinate X^{11}. This suggests that the doubled torus of string theory should generalise to a space with coordinates X^i on the T^n and coordinates $\widetilde{X}_{ij} = -\widetilde{X}_{ji}$ conjugate to the membrane wrapping charge Z^{ij}. For $n \geq 5$, further coordinates will be needed corresponding to the other M-theory brane charges, e.g. coordinates X_{ijklm} conjugate to 5-brane wrapping modes. For string theory, there is an explicit representation of the theory with the enlarged structure as a world-sheet theory allowing a concrete formulation, but for the M-theory generalisations, the absence of an analogue of the world-sheet formulation of the theory means that the discussion in that case has to be done in the context of an effective target space theory.

For example, for $n = 4$ the string theory doubled torus T^6 generalises to a torus T^{10} with four coordinates consisting of the T^4 coordinates X^i and six coordinates $\widetilde{X}_{ij}$ conjugate to membrane wrapping modes on T^4. The $\mathrm{O}(3,3;\mathbb{Z})$ or $\mathrm{SL}(4,\mathbb{Z})$ string theory T-duality symmetry is enlarged to the $\mathrm{SL}(5,\mathbb{Z})$ U-duality symmetry. The physical space is a T^4 slice of the T^{10}, and U-duality changes the physical subspace. The U-fold is represented as a T^{10} bundle over a 7-dimensional base space N, with $\mathrm{SL}(5,\mathbb{Z})$ transition functions, with the coordinates transforming according to the 10-dimensional representation of $\mathrm{SL}(5)$. Space time emerges from choosing a T^4 subspace of each T^{10} fibre, and for a geometric space with transition functions in a $\mathrm{GL}(4,\mathbb{Z})$ subgroup of $\mathrm{SL}(5,\mathbb{Z})$, this corresponds to a space time which is a T^4 bundle over N. For more general U-folds with non-geometric transition functions, there is a local slice $U_\alpha \times T^4$ of each patch $U_\alpha \times T^{10}$, but these do not fit together to form a manifold and there is no T^4 sub-bundle of the T^{10} bundle.

This is to be compared with the F'-theory constructions of [7], which involve a T^5 bundle over N. The $\mathrm{SL}(5,\mathbb{Z})$ transition functions can be used to construct either a T^5 bundle or a T^{10} bundle, but in the T^{10} construction discussed here, the extra dimensions play a physical role and are related to brane charges. In particular, non-trivial U-duality transition functions mean that there is no clear distinction between physical spacetime coordinates and auxiliary coordinates, and there can be configurations where each of the ten torus coordinates becomes a physical spacetime coordinate somewehere, but only four of the ten are simultaneously physical in any given local patch. Similar remarks apply to all the examples discussed here.

As a further example, consider M-theory on T^7. Viewed as IIA string theory on T^6, there is a doubled torus T^{12} and for M-theory this is enlarged to a T^{56} with 7 coordinates X^i on T^7, 21 coordinates X_{ij} conjugate to the M2-brane charge, 21 coordinates $X_{i_1 \ldots i_5}$ conjugate to the M5-brane charge, and 7 coordinates $X_{i_1 \ldots i_6}$ conjugate to the Kaluza-Klein monopole charge [45]. The enlarged spacetime would be a T^{56} bundle over a four dimensional base N with transition funtions in $E_7(\mathbb{Z})$. Physical space consists of choosing a local T^7 within each T^{56} fibre, and these will not fit together to form a T^7 bundle over spacetime in general unless the structure group is in the geometric group $\mathrm{GL}(7, \mathbb{Z})$ acting on the X^i.

In [46], it was suggested that spacetime can be pictured as arising as a surface in a larger space with a coordinate conjugate to each of the BPS brane charges in the superalgebra, with duality acting to change the embedding of the spacetime slice in the bigger space. The construction given here gives a concrete realisation of this in the context of perturbative string theory, and it seems to give the natural framework for discussing non-geometric backgrounds.

The discussion here has been concerned mainly with the case of spaces with a torus fibration, but can be generalised to those with a fibration by Calabi-Yau spaces or other spaces on which a stringy duality acts. In the case of a torus, the duality naturally leads to the doubling of the internal space in string theory, giving the picture of space-time as a surface in a larger space. It would be interesting to see to what extent this could be extended to more general spaces which do not have a torus or Calabi-Yau fibration and ask whether a doubled geometry, or the extension arising in M-theory, has a role to play in that case too. In [47], the possibility of non-geometric string backgrounds which do not have torus fibrations will be explored.

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